Asymptotic Convergence of Higher-Order Accurate Panel Methods

B. Oskam*
National Aerospace Laboratory NLR, Amsterdam, The Netherlands

Error analysis of incompressible potential flow solutions suggests that current panel methods employing piecewise quadratic doublet distributions, in combination with a Dirichlet boundary condition, can predict surface velocities within an error that vanishes proportionally to the third power of the panel size when panels are made arbitrarily small. This error analysis is based on computational experiments performed for a two-dimensional airfoil with a finite trailing-edge angle. The governing differential equation is Laplace's equation describing incompressible flow. The findings from the computational experiments are consistent with the observation that the corresponding low-order accurate panel methods, employing piecewise constant doublet distributions, yield a numerical approximation to the surface velocity that is accurate to first order in the panel size, when the size of the panels tends to zero.

Introduction

PANEL methods are useful tools to design airplane configurations in linearized potential flow. One way of formulating the integral equation is to prescribe internal Dirichlet boundary conditions. This approach has gained popularity over the years and has resulted in two distinct categories: low-order methods (e.g., Refs. 1-3) and higher-order methods (e.g., Refs. 4-9). All these methods employ doublet singularity distributions in combination with an internal Dirichlet boundary condition or offer the option to do so. One of the features distinguishing the two categories is the numerical representation of the doublet distribution: low-order methods use a piecewise constant distribution, while the higher-order approach is based on quadratic doublet representations[5-9] or a cubic representation.[4] The present study focuses on the evaluation of velocities by numerical differentiation; the alternative of evaluating the analogous integral representation of the velocity is not considered because it requires additional computation of aerodynamic influence coefficients. Using the doublet singularity formulation in combination with an internal Dirichlet boundary condition, it is well known that the surface velocity vector is given exactly by the gradient of the doublet distribution. Accordingly, in panel methods the surface velocity can be obtained by numerical differentiation. However, if one takes a closer look at the order of accuracy of this numerical differentiation, one may find a surprising result. The numerical differentiation in higher-order panel methods is summarized in Table 1.

From this table, one may observe that higher-order panel methods employ numerical differentiation, with accuracy one order lower than the doublet representation being differentiated. The quadratic doublet representations have a local truncation error of third order, while the corresponding surface velocity is obtained by second-order-accurate numerical differentiation. Thus, the higher-order panel methods obtain numerical solutions to the surface velocity with an order of accuracy that is expected to be one order lower than the truncation error of the local doublet representation.

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*Senior Research Engineer. Member AIAA.

The above procedures are in sharp contrast with common practice in low-order-accurate panel methods. Because these low-order-accurate methods employ piecewise constant doublet representations, with a local truncation error of first order, there is no room left to lower the order of accuracy of the numerical differentiation to obtain the surface velocity. As a matter of fact, it is found that low-order panel methods[1-3] obviously use numerical differentiation of at least first-order accuracy. The general observation is that this low-order approach—i.e., first-order-accurate doublet representations in combination with first-order-accurate numerical differentiation—yields numerical solutions of the surface velocity with an error of first order in the panel size when the panel size tends to zero. For example numerical experiments[10] conducted to reveal the global accuracy of a low-order panel method, when applied to von Kármán-Trefftz profiles with finite trailing-edge angles, suggest that the global accuracy is first order in the panel size. This global accuracy is defined as the norm of the difference between the numerical solution of the surface velocity and the exact solution.

Thus, the whole situation is quite surprising. Low-order panel methods with piecewise uniform doublet distributions display a global accuracy of first order, while higher-order panel methods with piecewise quadratic doublet distributions are not expected to have a corresponding global accuracy of third order, as is evident from the widespread usage of second-order-accurate numerical differentiation to obtain the surface velocity. The present paper addresses some aspects of this anomalous situation by presenting the results of computational experiments conducted with a higher-order panel method.

Higher-Order Panel Method

The particular higher-order panel method employed is based on the well-known integral representation of a potential induced by a doublet singularity distribution on a two-dimensional contour. In the absence of source singularities, the external tangential flow boundary condition is identical with the internal Dirichlet boundary condition that requires the total velocity potential to be zero on the inside of the contour. A standard integral equation is obtained by substitution of the integral representation into the Dirichlet boundary condition. The boundary value problem is completed by adding a Kutta condition requiring the upper and lower side limiting values of the tangential velocity components at the trailing edge to be identical.

Table 1 Distinguishing features of higher-order panel methods comprising doublet distributions and internal Dirichlet boundary conditions

Originator	Panel geometry	Doublet representation	AIC computation	Surface velocity
Botta[4]	Cubic spline	Cubic spline	Gaussian rules of order 6, 10, or 16	Derivative of cubic spline
Bristow MCAIR[7]	Flat	Piecewise quadratic	Exact	Second-order-accurate finite difference
Labrujere[5]	Quadratic spline	Quadratic spline	Small curvature expansion	Derivative of quadratic spline
Lötstedt[9]	Flat or quadratic	Piecewise constant or piecewise quadratic	Exact or small curvature expansion	Second-order-accurate finite difference
Magnus and Epton (PAN AIR,[6])	Each panel divided into 8 flat subpanels	Piecewise quadratic	Exact plus far-field expansions	Second-order-accurate finite difference
Oskam[8]	Quadratic	Piecewise quadratic	Small curvature expansion	Second-order-accurate finite difference

Let the contour of an airfoil be divided in N panels, and let the panel size be denoted by $h_i\,(i=2,3,\ldots,N+1)$. The locally quadratic doublet representation is given by

$$\tilde{\mu}(\xi)=\mu_i+(\mathrm{d}\mu/\mathrm{d}\xi)_i(\xi-\xi_i)+(\mathrm{d}^2\mu/\mathrm{d}\xi^2)_i\tfrac{1}{2}(\xi-\xi_i)^2 \quad (1)$$

$$\text{for } |\xi-\xi_i|\le h_i \text{ and } i=2,3,\ldots,N+1$$

where ξ represents a sufficiently accurate representation of the distance measured along the contour of the airfoil section. For convenience of notation we define

$$\xi_1=0 \qquad \xi_2=\tfrac{1}{2}h_2$$

$$\xi_{i+1}=\xi_i+\tfrac{1}{2}(h_i+h_{i+1}) \text{ for } (i=2,3,\ldots,N)$$

$$\xi_{N+2}=\xi_{N+1}+\tfrac{1}{2}h_{N+1} \quad (2)$$

where $[\xi_1,\ \xi_{N+2}]$ denotes the integration interval and where $\xi_i\,(i=2,3,\ \ldots,N+1)$ are the midpoints of the panels.

For the purpose of studying the global accuracy of higher-order panel methods with quadratic doublet representations, a five-point centered difference approximation to the first and second derivative is used. These approximations can be written as

$$(\mathrm{d}\mu/\mathrm{d}\xi)_i=\sum_{j=j_{\min}}^{j_{\max}} B_{ij}\mu_{i+j} \qquad i=2,\ldots,N+1 \quad (3a)$$

$$(\mathrm{d}^2\mu/\mathrm{d}\xi^2)_i=\sum_{j=j_{\min}}^{j_{\max}} C_{ij}\mu_{i+j} \qquad i=2,\ldots,N+1 \quad (3b)$$

where

$$j_{\min}=\max(-2,1-i)$$

$$j_{\max}=\min(2,2+N-i) \quad (3c)$$

The limiters $j_{\min}$ and $j_{\max}$, defined in Eq. (3c), reduce the five-point approximations to four point difference approximations at the first and last panels of the integration contour. The parameters $\mu_i\,(i=1,2,\ldots,N+2)$ are the function values of the doublet representations at $\xi_i\,(i=1,2,\ldots,N+2)$.

The actual difference formulas are derived using a $(j_{\max}-j_{\min}+1)$-point interpolation polynomial written in product notation,

$$P_i(\xi)=\sum_{j=j_{\min}}^{j_{\max}} \mu_{i+j}\prod_{\substack{k=j_{\min}\\k\neq j}}^{j_{\max}}\frac{\xi-\xi_{i+k}}{\xi_{i+j}-\xi_{i+k}} \quad (4)$$

for $i=2,3,\ldots,N+1$. Differentiation of P_i and evaluation at ξ_i $(i=2,3,\ldots,N+1)$ yields

$$B_{ij}=\sum_{\substack{l=j_{\min}\\l\neq j}}^{j_{\max}}\left\{\prod_{\substack{k=j_{\min}\\k\neq j\\k\neq l}}^{j_{\max}}(\xi_i-\xi_{i+k})\right\}\Bigg/\left\{\prod_{\substack{k=j_{\min}\\k\neq j}}^{j_{\max}}(\xi_{i+j}-\xi_{i+k})\right\} \quad (5a)$$

and

$$C_{ij}=\sum_{\substack{m=j_{\min}\\m\neq j}}^{j_{\max}}\left[\sum_{\substack{l=j_{\min}\\l\neq j\\l\neq m}}^{j_{\max}}\left\{\prod_{\substack{k=j_{\min}\\k\neq j\\k\neq l\\k\neq m}}^{j_{\max}}(\xi_i-\xi_{i+k})\right\}\right] \div\left\{\prod_{\substack{k=j_{\min}\\k\neq j}}^{j_{\max}}(\xi_{i+j}-\xi_{i+k})\right\} \quad (5b)$$

for $i=2,3,\ldots,N+1$, and $j=j_{\min},\ldots,j_{\max}$

In the case of a curved contour, such as an airfoil section, curved panels are employed. The present piecewise quadratic approximation of the parametric representation of the geometry is continuous over panel edges. These quadratic elements (curved panels) are combined with a small curvature expansion of the integrals for the aerodynamic influence coefficients, such that the geometry approximation is consistent with the piecewise quadratic doublet approximation[11].

The system of linear equations is derived from prescribing the internal potential at N collocation points $\xi_i(i=2,3,\ldots,N+1)$, i.e., at panel midpoints, while the value of the doublet representation at ξ_1 is equated to zero without any loss of generality. The equation expressing the Kutta condition equates the upper and lower side limiting values of the tangential velocity at the trailing edge. The discrete values of the velocity at the upper and lower side of the trailing edge are obtained by extrapolation of velocities in panel midpoints, Eq. (3a), toward the trailing edge along upper and lower side. The resulting Kutta equation can be written as

$$\sum_{i=2}^{1+p} a_i (\mathrm{d}\mu/\mathrm{d}\xi)_i + \sum_{i=N+2-p}^{N+1} a_i (\mathrm{d}\mu/\mathrm{d}\xi)_i = 0 \tag{6a}$$

for a p-point extrapolation formula. The present two-point extrapolation is given by

$$a_2 = \frac{2h_2 + h_3}{h_2 + h_3} \qquad a_3 = \frac{-h_2}{h_2 + h_3}$$

$$a_{N-1} = \frac{-h_{N+1}}{h_{N+1} + h_N} \qquad a_N = \frac{2h_{N+1} + h_N}{h_{N+1} + h_N} \tag{6b}$$

In summary, we have N Dirichlet conditions plus one Kutta condition, i.e., $N+1$ equations, for $N+1$ unknown doublet parameters, $\mu_i \, (i=2,3,\ldots,N+2)$.

Numerical Results

An example of this higher-order panel method applied to a 12%-thick von Kármán-Trefftz airfoil with a trailing-edge angle of 15 deg (KT0012) is shown in Fig. 1. This airfoil, at an angle of attack of 10 deg, is used as a test case to analyze the global accuracy of the numerical solutions obtained with the higher-order panel method. This global accuracy is defined in terms of finite-dimensional norms L_1, L_2, and L_∞ of the errors in the surface velocity at the collocation points $\xi_i \, (i=2,3, \ldots, N+1)$. Let the exact velocity at these points be denoted by $q_i \, (i=2,3,\ldots,N+1)$, and let the numerical solution be denoted by q_i^h, and let it be obtained from Eq. (3a)

$$q_i^h = \sum_{j=j_{\min}}^{j_{\max}} B_{ij} \mu_{i+j} \qquad i=2,3,\ldots,N+1 \tag{7}$$

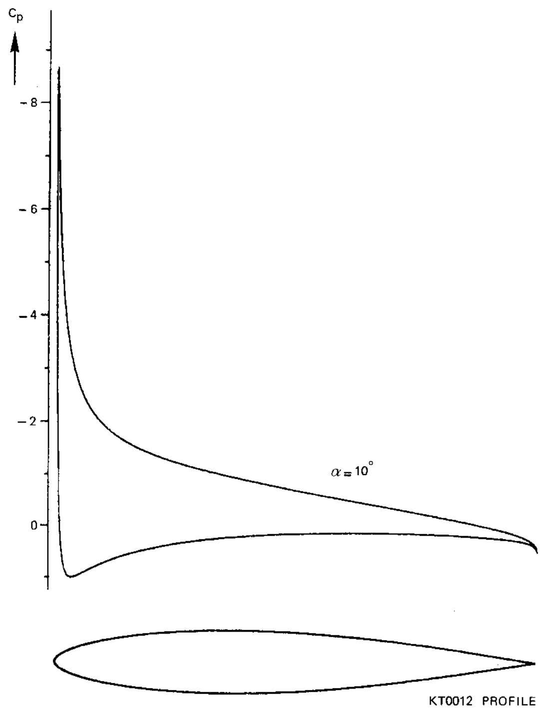

Fig. 1 Distribution of presure coefficient C_p for a 12%-thick von Kármán-Trefftz profile at 10-deg angle of attack.

where the vector μ_i is the solution of the linear system. Then, define the error measures according to

$$E_1 = \left\{\sum_i |q_i - q_i^h|\right\} \Big/ \left\{\sum_i |q_i|\right\} \tag{8a}$$

$$E_2 = \left[\left\{\sum_i (q_i - q_i^h)^2\right\} \Big/ \left\{\sum_i q_i^2\right\}\right]^{1/2} \tag{8b}$$

$$E_\infty = \left\{\max_i |q_i - q_i^h|\right\} \Big/ \left\{\max_i |q_i|\right\} \tag{8c}$$

where the index i ranges from $i=2$ up to $i=N+1$, but the summations in Eqs. (8a) and (8b) and the selection in Eq. (8c) exclude the points within a small radius of the trailing edge. This radius is 1% of the airfoil chord in the present examples. This local exclusion in the norms, Eqs. (8a-8c), is introduced because the numerical solutions are not intended to approximate the singular behavior near the trailing edge. These error measures, E_1, E_2, and E_∞ express the global accuracy of the numerical solution in terms of the magnitude of the exact solution in the same norm.

The results of numerical experiments for a sequence of 45 cases defined by the number of panels

$$N = 8,9,10,\ldots,20,22,24,\ldots,40,45,50,\ldots,100,110,120,\ldots,200$$

are given in Figs. 2a-2c for the KT0012 airfoil at 10-deg incidence. The outcome suggests that the global error measures E_1, E_2, and E_∞ satisfy the inequality

$$E \leq K(\bar{h})^3 \text{ for } \bar{h} \text{ sufficiently small}$$

where $\bar{h}$ is an average panel size, inversely proportional to N, and K is a constant independent of N. In summary, it is concluded that the solutions of the method described in the section entitled "Higher-Order Panel Method," employing a piecewise quadratic doublet distribution and sufficiently accurate numerical differentiation, in combination with an internal Dirichlet boundary condition, display third-order accuracy for sufficiently small panel size.

Discussion

The present solutions of Laplace's equation demonstrate that the higher-order approach with quadratic doublet representations can be brought into line with the low-order approach based on piecewise constant doublet singularity distributions. The asymptotic convergence to the exact solution is suggested to be third-order for the quadratic approach; this order of accuracy is consistent with the first-order asymptotic convergence of low-order methods based on a piecewise constant doublet representation. To achieve such a third-order accuracy in the approximate surface velocity requires numerical differentiation with at least third-order accuracy. This requirement is clearly a novel item in view of the widespread use of second-order-accurate numerical differentiation in combination with quadratic doublet representations (see Table 1).

The present five-point centered difference approximation used to obtain the velocity, Eq. (7), has a local truncation error of order h^4. This is one order higher than necessary in view of the third-order accuracy requirement above. This additional order does not affect the asymptotic convergence of the solutions, although simpler four-point differences will suffice to achieve global accuracy of third-order. Similar arguments apply to the difference approximations of the derivatives in the local quadratic doublet representation, Eq. (1), which has a local truncation error of order h^3. This means that the approach described in the section entitled "Higher-Order Panel Method" can be refined at a number of points without reducing the global accuracy.

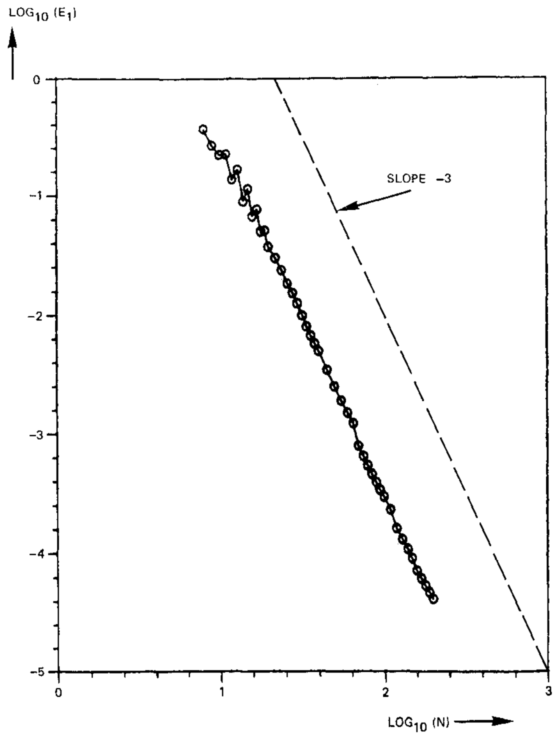

Fig. 2a Average error E_1 of the numerical solutions vs the number of panels N for a KT0012 profile at 10-deg incidence.

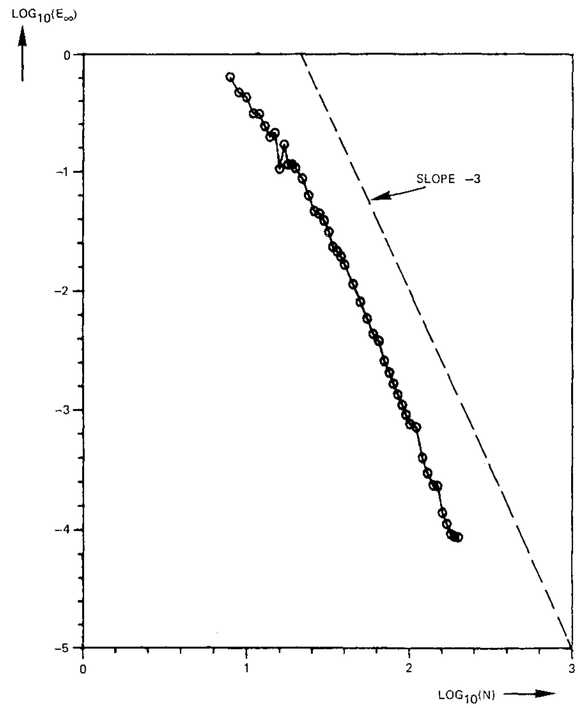

Fig. 2c Maximum-norm error measure E_∞ of the numerical solutions vs the number of panels N for a KT0012 profile at 10-deg incidence.

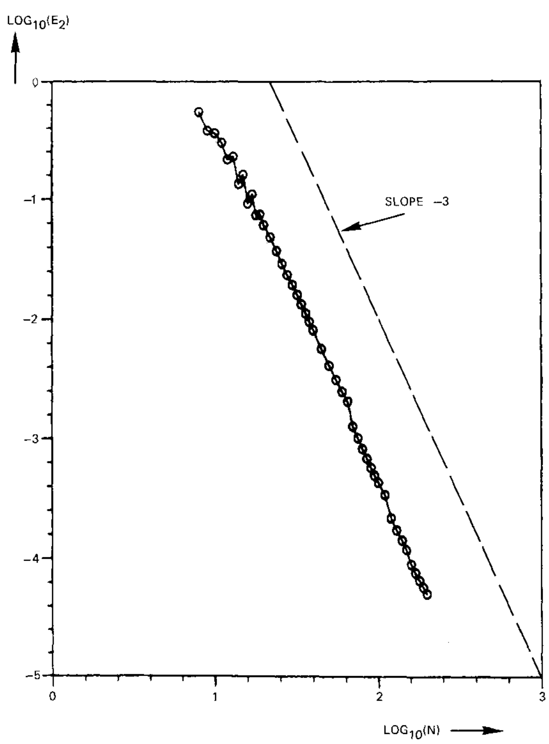

Fig. 2b Root-mean-square error measure E_2 of the numerical solutions vs the number of panels N for a KT0012 profile at 10-deg incidence.

The present results also suggest that a global accuracy requirement of second-order[12] can be satisfied with higher-order panel methods based on linear representations of the doublet distribution in combination with Dirichlet boundary conditions. Second-order accuracy can also be achieved by higher-order panel methods based on external Neumann boundary conditions (see Ref. 13 for the same test case). The linear doublet approach in combination with internal Dirichlet boundary conditions offers the prospect of achieving second-order accuracy with a simpler method, which appears to be attractive, especially in three dimensions. However, it should be borne in mind that the present computational experiments are just a necessary first step, and that they are performed for a two-dimensional rather than three-dimensional configuration. Functional analysis needs to be sought to place current panel methods on a theoretical foundation, such that the current findings can be explained in terms of a property of the integral operator involved.

Conclusions

There is not nearly enough information available about the accuracy of panel methods in general, and about methods based on internal Dirichlet conditions in particular. Present results of computational experiments indicate that this last category of higher-order panel methods with piecewise quadratic doublet distribution should be expected to display a global accuracy of third order. This order of accuracy brings the higher-order approach into line with low-order methods employing piecewise constant doublet distributions.

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